Journal of Engineering Mathematics 22: 225-238 (1988)

Calculation of forces and moments in vortex methods

J.J.W. VAN DER VEGT
Maritime Research Institute Netherlands (MARIN), P.O. Box 28, 6700 AA Wageningen, The Netherlands

Received 14 September 1987; accepted in revised form 10 March 1988

Abstract. An alternative formulation for the calculation of forces and moments acting on a body in a three-dimensional unsteady viscous incompressible flow field is derived. The formulation is especially useful when the Navier-Stokes equations are solved in vorticity formulation.

1. Introduction

Vortex methods for the description of viscous incompressible flows can provide an interesting alternative to the widely-used primitive-variable methods. Vortex methods determine the flow field by calculating the evolution of the vorticity field using a Lagrangian method. They are frequently applied to the calculation of vortex shedding from bluff bodies, see for instance van Oortmerssen et al. [1] and Stansby et al. [2]. In two dimensions vortex methods are relatively easy to apply, but a more extensive use is hindered by problems related to the accuracy, numerical efficiency and the extension to three dimensions of these methods. All these aspects are currently subject to intensive research, but one aspect that is especially important for engineering applications, the calculation of forces and moments, has received only limited attention. The problems with the calculation of forces and moments originate from the fact that in a vorticity formulation the pressure is eliminated, which is essential information for the calculation of forces and moments. The purpose of this article therefore is to discuss an alternative method to compute forces and moments without requiring the calculation of the pressure and more suited to vortex methods than the formulations used for primitive variable algorithms. Due to the fact that the algorithm for the computation of forces and moments in a viscous fluid is strongly related to the vortex method, first some basic aspects of these methods will be discussed shortly.

Vortex methods for viscous flows solve the viscous vorticity transport equation, which is obtained by taking the curl of the Navier-Stokes equations. They are based on the observation that large eddies outside the close vicinity of the wall in a slightly viscous fluid behave approximately as if the fluid were inviscid. This phenomenon was used by Chorin [3] to extend the vortex methods for inviscid flows to viscous flows. The simulation of a viscous flow was accomplished by subsequently calculating for small time intervals the evolution of the vorticity field neglecting viscosity, followed by giving the vortices a random displacement, with a Gaussian probability density, in order to take care of the effects of viscosity. This process is completed each time step by the creation of new vorticity at the body surface in order to satisfy the no-slip condition, because during the inviscid step only the no-flux condition is satisfied at the body surface. This method initially obtained a lot of criticism, Milinazzo et al. [4], but Beale et al. [5] demonstrated its validity in an unbounded fluid.

Recently van der Vegt [6] proved convergence of the random-vortex method to solutions of the viscous vorticity-transport equation for three-dimensional viscous incompressible flows around bodies without sharp edges, using a variational method.

The vortex method for viscous flows thus strongly depends on an efficient algorithm for inviscid flows. In the first attempts to use vortex methods for the solution of two-dimensional inviscid flows the vorticity field was represented by a set of point vortices which are transported with the local fluid velocity. Although the point-vortex method exactly satisfies the inviscid vorticity transport equation the method is not valid for a continuous vorticity field. When two vortices come close together the numerical errors grow rapidly due to the singular behaviour of the induced velocities, finally resulting in chaotic motions of the vortices. The way to solve this problem is to represent the vorticity field by a finite number of vortices in which the vorticity is spread out over some smooth kernel with compact support, resulting in the vortex-blob method. This method has been subject of extensive theoretical research. Convergence of the vortex-blob method to solutions of the inviscid vorticity transport equation has been proved in both two and three dimensions by for instance Beale et al. [7–8] and Anderson et al. [9]. An additional advantage of the vortex-blob method is that the extension to three dimensions presents no special problems compared to the three-dimensional counterpart of the point-vortex method namely the vortex-line method. A numerical algorithm for the calculation of the three-dimensional flow field around a circular cylinder, with the correct incorporation of the vortex-stretching contribution, is presented by van der Vegt [6].

A second problem related to vortex methods is their low numerical efficiency. In order to compute the evolution of the vorticity field the mutually-induced velocities of the vortices must be calculated. In an unbounded fluid these induced velocities can be calculated using Biot and Savart's law of interaction. This computation has, however, an operation count proportional to the square of the number of point vortices or vortex blobs. This severely limits the maximum attainable number of vortices for a realistic flow simulation. There are two methods to surpass this limit: either the use of multiprocessors and multipole expansions as done by Baden [10] and Greengard et al. [11], or the use of fast algorithms for the solution of the Poisson equation using a grid fixed in space, as done for instance by Couet et al. [12] for three-dimensional mixing layers and van der Vegt [6] for both two- and three-dimensional flows around circular cylinders.

The essential features of a vortex model now can be summarised as follows: Let $\mathbf{R}(\mathbf{r}', t, t_0)$ be an invertible continuous differentiable flow map with a Jacobian determinant unity. This flow map represents the transformation of the initial fluid volume Ω_0 at time t_0 to the fluid volume Ω at time t. Then the position $\mathbf{r}$ at time t of a fluid particle initially at $\mathbf{r}'$ is given by the relation

$$\mathbf{r} = \mathbf{R}(\mathbf{r}', t, t_0). \tag{1.1}$$

The flow map $\mathbf{R}$ must have a Jacobian determinant unity due to the incompressibility constraint, see Serrin [13]. It represents the Lagrangian coordinate system, while the fluid-particle velocity is related to the curl of a divergence-free vector stream function $\mathbf{A}$, representing the Eulerian velocity field:

$$\dot{\mathbf{R}} = \nabla_{\mathbf{R}} \times \mathbf{A}(\mathbf{R}, t). \tag{1.2}$$

The vector stream function **A** is related to the vorticity field by the Poisson equation

$$\nabla^2 \mathbf{A} = -\boldsymbol{\omega}(\mathbf{r}, t) \tag{1.3}$$

$$= -\boldsymbol{\omega}_0(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \mathbf{R}(\mathbf{r}', t, t_0), \tag{1.4}$$

where the right-hand sides of this expression represent the relation between the vorticity field in the Eulerian and Lagrangian systems. When this set of equations is supplied with an initial vorticity field $\boldsymbol{\omega}_0$, satisfying the boundary condition $\mathbf{n} \cdot \boldsymbol{\omega}_0 = 0$ at $\partial\Omega_0$, together with the no-flux condition $\mathbf{n} \cdot (\nabla \times \mathbf{A}) = 0$ at the body surface, with $\mathbf{n}$ the unit outward normal vector, then these equations are equivalent with the inviscid vorticity-transport equation. This set of equations was put in a variational formulation using Lagrangian and Hamiltonian field theory by van der Vegt [6], which is a useful tool to obtain a numerical algorithm.

The vorticity field in a vortex-blob method now is represented by

$$\boldsymbol{\omega}^*(\mathbf{r}, t) = \sum_j \int_\Omega f_\sigma(\mathbf{r} - \mathbf{s} - \mathbf{r}_j)\boldsymbol{\omega}(\mathbf{s}, t)\, \mathrm{d}s, \tag{1.5}$$

where the function f_σ, the vorticity distribution in the vortex blob, is common to all vortices and σ represents the core radius of a vortex with position $\mathbf{r}_j$.

The vortex method for a viscous flow can now be constructed using the algorithm for inviscid flow as discussed earlier. A precise formulation in terms of a product algorithm can be found in van der Vegt [6]. However, the essential features of a vortex method which are necessary for the calculation of forces and moments are that any vortex method must solve a Poisson equation and represents the vorticity field as a finite sum of vortices with a compact support.

As already mentioned, the problems with the calculation of forces and moments when using vortex methods originate from the fact that the pressure is eliminated in a vorticity formulation. In order to be able to determine the forces and moments in this case some authors, for instance Graham [14] and Deffenbaugh et al. [15], used the so-called Blasius equation. The use of the Blasius equation is, however, not valid for viscous flows and the method can only be applied in two dimensions. Another approach is to obtain the pressure directly from an integration of the tangential derivative of the pressure at the body surface, which in two dimensions is given by

$$\frac{\partial p}{\partial s} = -\mu \frac{\partial \omega}{\partial n}, \tag{1.6}$$

with s the coordinate along the surface and μ the viscosity coefficient. This method was used, for instance, by Chorin [3] and Stansby et al. [2]. In order to obtain the forces and moments one has to integrate twice, first to obtain the pressure, next to obtain forces and moments. Numerical experiments, performed by Stansby et al. [2], however, showed a strong dependency on the arbitrary initial position of the integration which should not exist.

It is, however, possible to perform this integration analytically and to relate the forces and moments directly to the vorticity and velocity field, instead of an integral of the normal derivative of the vorticity field. In the next part of this article this method will be discussed together with its implementation in vortex methods. The alternative method to compute forces and moments is based on a projection of the Navier–Stokes equations on a space spanned by a set of functions satisfying Laplace's equation and choosing appropriate boundary conditions for these functions.

2. Calculation of forces and moments in a viscous incompressible fluid using vortex methods

The algorithm for the calculation of forces and moments suited to vortex methods is derived directly from the Navier–Stokes equations in primitive-variable formulation. However, prior to the derivation of an expression for the integral of the pressure at the body surface, first some remarks are necessary about the velocity and pressure field. The total velocity field $\mathbf{u}^*$ must satisfy, apart from the Navier–Stokes equations, the incompressibility constraint $\operatorname{div} \mathbf{u}^* = 0$. In addition it also has to approach a uniform velocity field $\mathbf{U}(t)$ at great distance from a body together with the no-slip condition $\mathbf{u}^* = \mathbf{u}_w$ at the body surface $S \subset \partial\Omega$, with $\mathbf{u}_w$ the velocity of the body surface S and Ω the fluid domain. In order to ease the calculations the velocity and pressure fields $\mathbf{u}^*$ and p^* are separated into two components,

$$\mathbf{u}^*(\mathbf{r}, t) = \mathbf{u}(\mathbf{r}, t) + \mathbf{U}(t), \tag{2.1}$$

$$p^*(\mathbf{r}, t) = p(\mathbf{r}, t) + P(t), \tag{2.2}$$

with $\mathbf{U}(t)$ and $P(t)$ the velocity and pressure field of the undisturbed flow and $\mathbf{u}$ and p the disturbance velocity and pressure field. It is assumed that the fields $\mathbf{u}$ and ∇p are both elements of $L^2(\Omega)$, with $L^2(\Omega)$ the Hilbert space with inner product $\langle \mathbf{f} | \mathbf{g} \rangle = \int_\Omega \mathbf{f} \cdot \mathbf{g} \, d\Omega$.

The equations for the disturbance velocity and pressure field then are obtained by subtracting the equations for the undisturbed field from the Navier-Stokes equations and using the decomposition given by equations (2.1) and (2.2):

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} + \mathbf{U}) \cdot \nabla \mathbf{u} = -\frac{1}{\varrho} \nabla p - \nu \nabla \times (\nabla \times \mathbf{u}), \tag{2.3}$$

with ϱ the density and ν the kinematic viscosity of the fluid. The no-slip condition for the disturbance velocity field at the body surface S then changes into

$$\mathbf{u} = \mathbf{u}_w - \mathbf{U}(t), \tag{2.4}$$

while at great distance from the body both $\mathbf{u}$ and ∇p must approach a zero value.

An expression for the integral of the pressure at the body surface S now can be obtained by defining an additional set of test functions η belonging to the vector space L^2 and satisfying Laplace's equation. They form the subspace H_1 of L^2 defined as

$$H_1 = \{\mathbf{v} \in L^2 \,|\, \mathbf{v} = \nabla \eta, \nabla^2 \eta = 0\}. \tag{2.5}$$

The basic step in the elimination of the pressure from the formulation of forces and moments acting on a body now consists of projecting the space of divergence-free vector fields, the

Hilbert space H, a subspace of L^2, onto the subspace H_1 using the standard L^2 inner product. Quartapelle et al. [16] used this approach to derive some kind of equivalent formulation, but their result has some serious deficiencies, namely:

(i) In their formulation they end up with a volume integral, $\int \mathbf{u} \cdot \nabla \mathbf{u} \, \mathrm{d}V$, which is difficult to evaluate and requires the determination of the gradient of the velocity field which is not known when the Navier–Stokes equations are solved with a Lagrangian vortex method.

(ii) The contribution of the fluid acceleration to the forces and moments requires a cumbersome evaluation of an integral at a surface tending to infinity.

The algorithm presented in this article overcomes these problems if the flow field is determined using a vortex method.

In the next part of this section, first an expression for the integral of the pressure at the surface of a body in terms of the velocity and vorticity field will be derived which then will be used to eliminate the pressure from the formulation of the forces and moments.

The projection of the momentum equations, equation (2.3), onto the space H_1 using the standard inner product of the Hilbert space L^2 results in the following volume integral:

$$-\frac{1}{\varrho}\int_{\Omega(t)} \nabla p \cdot \nabla \eta \, \mathrm{d}V = \int_{\Omega(t)} \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} + \mathbf{U}) \cdot \nabla \mathbf{u} + \nu \nabla \mathbf{x} (\nabla \mathbf{x} \mathbf{u}) \right\} \cdot \nabla \eta \, \mathrm{d}V. \tag{2.6}$$

Here $\Omega(t)$ is the time-dependent fluid domain with respect to an inertial coordinate system.

With some well-known vector relations this expression can be simplified using the fact that the velocity field is solenoidal and the function η satisfies the Laplace equation:

$$-\frac{1}{\varrho}\int_{\Omega(t)} \nabla \cdot (p \nabla \eta) \, \mathrm{d}V = \int_{\Omega(t)} \left\{ \nabla \cdot \left[\eta \frac{\partial \mathbf{u}}{\partial t} + \nu(\omega \times \nabla \eta) + \tfrac{1}{2}((\mathbf{u} + \mathbf{U}) \cdot (\mathbf{u} + \mathbf{U})) \nabla \eta \right] - ((\mathbf{u} + \mathbf{U}) \times \omega) \cdot \nabla \eta \right\} \mathrm{d}V. \tag{2.7}$$

The volume integrals of this inner product can be partly reduced by Gauss' theorem into surface integrals on S, the body surface, and S_∞, a sphere tending to infinity with its centre, the centre of gravity of the body. The surface S_∞ is continually moving with the body, so the volume Ω between S and S_∞ is constant (see Fig. 1).

Gauss' theorem applied to the volume Ω now yields

$$\begin{aligned} -\frac{1}{\varrho}\int_{S \cup S_\infty} p \mathbf{n} \cdot \nabla \eta \, \mathrm{d}S = & \int_{S \cup S_\infty} \left\{ \eta \left(\mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) + \nu(\mathbf{n} \times \omega) \cdot \nabla \eta \right. \\ & \left. + \tfrac{1}{2}((\mathbf{u} + \mathbf{U}) \cdot (\mathbf{u} + \mathbf{U}))(\mathbf{n} \cdot \nabla \eta) \right\} \mathrm{d}S \\ & + \int_\Omega ((\mathbf{u} + \mathbf{U}) \times \omega) \cdot \nabla \eta \, \mathrm{d}V; \end{aligned} \tag{2.8}$$

here $\mathbf{n}$ is the unit normal vector pointing into Ω.

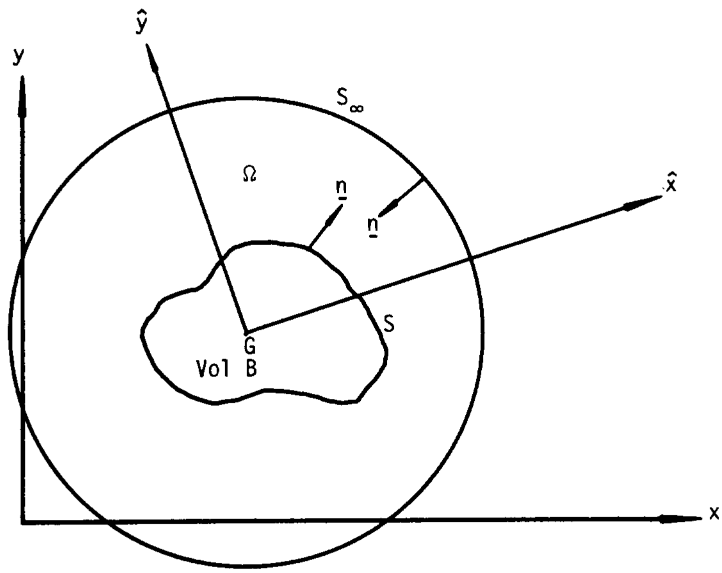

Fig. 1. Definition of geometry of the surfaces S and S_∞.

The expression for the disturbance pressure integrated at the body surface in terms of the velocity-vorticity field and an additional field η, equation (2.8), now is used to derive an alternative formulation for forces and moments acting on a body in a viscous incompressible flow.

The force acting on a body in a viscous flow field is usually defined as

$$\mathbf{F} = -\int_S \mathbf{t}^s \, \mathrm{d}S, \tag{2.9}$$

with $\mathbf{t}^s$ the Cauchy stress vector; see for instance Serrin [13]. By a result obtained by Berker, see Serrin [13], the stress vector at a fixed wall is equal to

$$\mathbf{t}^s = -p^*\mathbf{n} - \mu(\mathbf{n} \times \boldsymbol{\omega}), \tag{2.10}$$

with μ the viscosity coefficient, $\mathbf{n}$ the unit outward normal at S and $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}^*$ the vorticity field. Combining these results the force acting on a body in a viscous flow becomes equal to

$$\mathbf{F} = \int_S \{p^*\mathbf{n} + \mu(\mathbf{n} \times \boldsymbol{\omega})\} \, \mathrm{d}S. \tag{2.11}$$

After substitution of the decomposition into an undisturbed flow field and a disturbance flow field, equations (2.1) and (2.2), we obtain

$$\mathbf{F} = \int_S \{(P + p)\mathbf{n} + \mu(\mathbf{n} \times \boldsymbol{\omega})\} \, \mathrm{d}S, \tag{2.12}$$

using the assumption that the undisturbed flow field is irrotational. The first integral can be evaluated using Gauss' identity and the Navier–Stokes equations for the undisturbed field:

$$\int_S P\mathbf{n}\,\mathrm{d}S = \int_{\Omega_B} \nabla P\,\mathrm{d}V = -\varrho\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\,\mathrm{Vol}\,B \tag{2.13}$$

with $\Omega_B = \mathrm{Vol}\,B$ the volume of the body.

Thus the force acting on a body is equal to

$$\mathbf{F} = -\varrho\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\,\mathrm{Vol}\,B + \int_S \{p\mathbf{n} + \mu(\mathbf{n}\times\omega)\}\,\mathrm{d}S. \tag{2.14}$$

The surface integral of the disturbance pressure now can be eliminated using the inner product, equation (2.8), and imposing additional boundary conditions on the function η. To obtain time-independent boundary conditions for the function η, the disturbance flow field is considered with respect to a body-fixed Cartesian coordinate system with unit vectors $(\hat{\mathbf{e}}_{\hat{x}}, \hat{\mathbf{e}}_{\hat{y}}, \hat{\mathbf{e}}_{\hat{z}})$ with respect to the centre of gravity of the body, instead of an inertial coordinate system with unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. In the body-fixed coordinate system the surfaces S and S_∞ and the volume Ω are independent of time.

The component in the direction of the unit vector $\hat{\mathbf{e}}_{\hat{x}}$ of the force caused by the disturbance flow field is defined as

$$f_{\hat{x}} = \int_S \{\hat{p}\hat{\mathbf{n}} + \mu(\hat{\mathbf{n}}\times\hat{\omega})\}\cdot\hat{\mathbf{e}}_{\hat{x}}\,\mathrm{d}S. \tag{2.15}$$

By imposing the following conditions on the function $\hat{\eta}_{\hat{x}}$,

$$\nabla^2\hat{\eta}_{\hat{x}} = 0 \qquad \text{in } \Omega, \tag{2.16}$$

$$\hat{\mathbf{n}}\cdot\nabla\hat{\eta}_{\hat{x}} = -\hat{\mathbf{n}}\cdot\hat{\mathbf{e}}_{\hat{x}} \qquad \text{at } S, \tag{2.17}$$

$$\hat{\mathbf{n}}\cdot\nabla\hat{\eta}_{\hat{x}} = 0 \qquad \text{at } S_\infty, \tag{2.18}$$

the pressure in the formulation of the force can be eliminated using the inner product, equation (2.8). The force component $f_{\hat{x}}$ now becomes equal to

$$f_{\hat{x}} = \varrho\int_{S\cup S_\infty}\left\{\hat{\eta}_{\hat{x}}\left(\hat{\mathbf{n}}\cdot\frac{\partial\hat{\mathbf{u}}}{\partial t}\right) + \nu(\hat{\mathbf{n}}\times\hat{\omega})\cdot\nabla\hat{\eta}_{\hat{x}} + \tfrac{1}{2}((\hat{\mathbf{u}}+\hat{\mathbf{U}})\cdot(\hat{\mathbf{u}}+\hat{\mathbf{U}}))(\hat{\mathbf{n}}\cdot\nabla\hat{\eta}_{\hat{x}})\right\}\mathrm{d}S$$
$$+ \mu\int_S(\hat{\mathbf{n}}\times\hat{\omega})\cdot\hat{\mathbf{e}}_{\hat{x}}\,\mathrm{d}S + \varrho\int_\Omega((\hat{\mathbf{u}}+\hat{\mathbf{U}})\times\hat{\omega})\cdot\nabla\hat{\eta}_{\hat{x}}\,\mathrm{d}V; \tag{2.19}$$

^ refers to variables with respect to the body-fixed coordinate system. Introduction of the boundary conditions of $\hat{\eta}_{\hat{x}}$ in this equation yields

$$f_{\hat{x}} = \varrho\int_{S\cup S_\infty}\hat{\eta}_{\hat{x}}\left(\hat{\mathbf{n}}\cdot\frac{\partial\hat{\mathbf{u}}}{\partial t}\right)\mathrm{d}S + \mu\int_S(\hat{\mathbf{n}}\times\hat{\omega})\cdot(\nabla\hat{\eta}_{\hat{x}} + \hat{\mathbf{e}}_{\hat{x}})\,\mathrm{d}S$$
$$+ \varrho\int_\Omega((\hat{\mathbf{u}}+\hat{\mathbf{U}})\times\hat{\omega})\cdot\nabla\hat{\eta}_{\hat{x}}\,\mathrm{d}V. \tag{2.20}$$

This result was obtained because the contribution of the integral

$$\mu \int_{S_\infty} (\hat{\mathbf{n}} \times \hat{\omega}) \cdot \nabla \hat{\eta}_{\hat{x}} \, \mathrm{d}S \tag{2.21}$$

is zero due to the assumption that the undisturbed flow field $\mathbf{U}$ only depends on time and the asymptotic behaviour of $\hat{\omega}$ and $\nabla \hat{\eta}_{\hat{x}}$ for large values of r, the distance from the centre of gravity.

In addition, the contribution of the integral

$$\frac{1}{2} \int_{S \cup S_\infty} ((\hat{\mathbf{u}} + \hat{\mathbf{U}}) \cdot (\hat{\mathbf{u}} + \hat{\mathbf{U}}))(\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_{\hat{x}}) \, \mathrm{d}S \tag{2.22}$$

disappears at S and S_∞ by the boundary condition for $\hat{\eta}_{\hat{x}}$ at S_∞ and the no-slip condition at the wall, viz.

$$\hat{\mathbf{u}} + \hat{\mathbf{U}} = \hat{\mathbf{u}}_W = 0 \quad \text{at } S. \tag{2.23}$$

A similar result is valid for the $\hat{y}$ and $\hat{z}$ components of the force, however, with $\hat{x}$ replaced by $\hat{y}$, respectively $\hat{z}$. Combining the various contributions of the undisturbed flow field to the forces and transforming the latter to the inertial coordinate system we obtain the following new formulation for the forces in a viscous and incompressible flow field:

$$\begin{aligned} \mathbf{F} = {} & -\varrho \dot{\mathbf{U}} \operatorname{Vol} B + \varrho B(\dot{\boldsymbol{\alpha}}_g \times \mathbf{U} - \dot{\mathbf{U}}) - G(\dot{\boldsymbol{\alpha}}_g \times \dot{\mathbf{x}}_g - \ddot{\mathbf{x}}_g) \\ & + \mu T(\boldsymbol{\alpha}_g) \int_S ((\hat{\mathbf{n}} \times \hat{\omega}) \cdot \nabla \hat{\boldsymbol{\eta}} + (\hat{\mathbf{n}} \times \hat{\omega})) \, \mathrm{d}S \\ & + T(\boldsymbol{\alpha}_g) \sum_j \int_{\Omega_j} \{(\hat{\mathbf{u}}(\mathbf{r}, t) + \hat{\mathbf{U}}(t)) \times \hat{\omega}^*(\mathbf{r} - \mathbf{r}_j, t)\} \cdot \nabla \hat{\boldsymbol{\eta}} \, \mathrm{d}V \end{aligned} \tag{2.24}$$

with the matrices B and G defined as

$$B_{ij} = \int_S \hat{\eta}_i \hat{n}_j \, \mathrm{d}S, \qquad G_{ij} = \frac{1}{3} \int_S (\hat{\eta}_i \hat{n}_j - \hat{b}_i \hat{s}_j) \, \mathrm{d}S, \tag{2.25}$$

where the indices $i, j = \{1, 2, 3\}$ refer to the $\hat{x}, \hat{y}$ or $\hat{z}$ components of the vectors respectively. Here the vector $\dot{\boldsymbol{\alpha}}_g$ represents the angular velocity of the body, the vector $\mathbf{x}_g$ the position of the centre of gravity of the body and a dot means differentiation with respect to time. The matrix $T(\boldsymbol{\alpha}_g)$ gives the relation between the inertial and body-fixed coordinate systems and is defined as

$$T(\boldsymbol{\alpha}_g) = \begin{pmatrix} (\mathbf{e}_x \cdot \hat{\mathbf{e}}_{\hat{x}}) & (\mathbf{e}_x \cdot \hat{\mathbf{e}}_{\hat{y}}) & (\mathbf{e}_x \cdot \hat{\mathbf{e}}_{\hat{z}}) \\ (\mathbf{e}_x \cdot \hat{\mathbf{e}}_{\hat{y}}) & (\mathbf{e}_y \cdot \hat{\mathbf{e}}_{\hat{y}}) & (\mathbf{e}_y \cdot \hat{\mathbf{e}}_{\hat{z}}) \\ (\mathbf{e}_x \cdot \hat{\mathbf{e}}_{\hat{z}}) & (\mathbf{e}_y \cdot \hat{\mathbf{e}}_{\hat{z}}) & (\mathbf{e}_z \cdot \hat{\mathbf{e}}_{\hat{z}}) \end{pmatrix}, \tag{2.26}$$

where the innerproducts are equal to the cosines of the angles between the inertial and body-fixed coordinate systems, as given by the vector $\boldsymbol{\alpha}_g$. Finally the vectors $\hat{\eta}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{s}}$ are defined as $\hat{\boldsymbol{\eta}} = (\hat{\eta}_{\hat{x}}, \hat{\eta}_{\hat{y}}, \hat{\eta}_{\hat{z}})^T$; $\hat{\mathbf{b}} = -\hat{\mathbf{n}}$, where the superscript T means the transposed of a vector, and $\hat{\mathbf{s}}$ is a vector from the centre of gravity of the body to its surface S.

In the derivation of equation (2.24) the representation of the vorticity field as a set of vortex blobs with a compact support is used, as given by equation (1.5). The volume integral in equation (2.20) then can be transformed into the sum of volume integrals over the small neighbourhood Ω_j of the vortex blob with position $\mathbf{r}_j$.

The transformation of the integral at the surfaces S and S_∞ of the fluid acceleration, the first contribution on the right-hand side of equation (2.20), into the more practical integration at the body surface as shown in equation (2.25) is further discussed in the Appendix.

The expression for the forces on a body in a viscous fluid as given by equation (2.24) consists of three main contributions, namely forces related to the acceleration of fluid or body, which do not depend on viscosity, forces due to the viscosity of the fluid and finally forces caused by the vorticity in the fluid. It is interesting to note that the force on a body caused by a vortex blob is orthogonal to both the local fluid velocity and vorticity vector.

The derivation of the expression for the moments on a body in a viscous fluid is completely analogous to the derivation of the forces as discussed in the previous part. The expression for the moment is directly obtained by changing the vectors $\hat{\boldsymbol{\eta}}$ and $\hat{\mathbf{b}}$ into $\hat{\boldsymbol{\eta}}^M$ and $\hat{\mathbf{b}}^M$ and the replacement of the first term on the right-hand side of equation (2.24) by $\varrho\dot{\mathbf{U}} \times \mathbf{I}(t)$, with $\mathbf{I}(t)$ the area-moment vector of the body with respect to the centre of gravity at time t. The vector $\hat{\boldsymbol{\eta}}^M = (\hat{\eta}^M_{\hat{x}}, \hat{\eta}^M_{\hat{y}}, \hat{\eta}^M_{\hat{z}})$ for the moment must satisfy, analogous to the vector $\hat{\boldsymbol{\eta}}$ for the forces, the Laplace equation (2.16) and the boundary condition (2.18) for each component, while its boundary condition at the surface S is changed into

$$\mathbf{n} \cdot \nabla \hat{\boldsymbol{\eta}}^M = -(\hat{\mathbf{s}} \times \hat{\mathbf{n}}) = \hat{\mathbf{b}}^M. \tag{2.27}$$

The formulations for $\hat{\mathbf{F}}$ and $\hat{\mathbf{M}}$ given so far are non-unique at first sight. Due to the fact that the functions $\hat{\eta}$ only have to satisfy Neumann boundary conditions at the surfaces S and S_∞ the functions $\hat{\eta}$ are undetermined up to arbitrary constants $\hat{\eta}_{p\infty}$. It can be shown that these constants do not contribute to the forces and moments. In all but the inertia contribution only the gradient of the functions $\hat{\eta}$ is used to compute the forces and moments, thus removing the constants. The constants in the inertial contribution disappear using Gauss' identity:

$$\int_{S \cup S_\infty} \hat{\eta}_{p\infty} \left(\hat{\mathbf{n}} \cdot \frac{\partial \hat{\mathbf{u}}}{\partial t}\right) \mathrm{d}S = \hat{\eta}_{p\infty} \int_\Omega \operatorname{div} \hat{\mathbf{u}} \,\mathrm{d}V = 0 \tag{2.28}$$

and are zero because of the continuity equation.

The practical implementation of the calculation of forces and moments using equation (2.24) now consists of three steps:

(i) The calculation of the velocity and vorticity field using a vortex method as discussed in the introduction. Numerical algorithms for both two- and three-dimensional flows can be found for instance in van der Vegt [6].

(ii) The determination of the vectors $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\eta}}^M$ for the forces and moments respectively. Due to the fact that $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\eta}}^M$ are defined in a body-fixed coordinate system they have to be calculated only once. For simple geometries such as a cylinder or a sphere this can be done analytically. In the case of bodies of arbitrary shape it is beneficial to use the algorithm for the solution of the Poisson equation (1.3) which is available in any vortex method.

In two dimensions one has to determine three functions $\hat{\eta}$, respectively $\hat{\eta}_{\hat{x}}$, $\hat{\eta}_{\hat{y}}$ and $\hat{\eta}_{\hat{z}}^{M}$, related to the $\hat{x}$- and $\hat{y}$-component of the force and the $\hat{z}$-component of the moment. Here it is assumed that the $\hat{z}$-component is orthogonal to the two-dimensional plane. In two dimensions therefore three Laplace equations must be solved. In order to be able to use the algorithm for the solution of a Poisson equation (1.3), which reduces to a scalar Poisson equation in two dimensions, it is advisable to transform the Laplace equations for $\hat{\eta}$ with inhomogeneous boundary conditions into Poisson equations with homogeneous boundary conditions.

Define the function $\hat{\eta}_{\hat{x}}$ as

$$\hat{\eta}_{\hat{x}} = \hat{\psi}_{\hat{x}} - \hat{x}H(\tfrac{1}{2}D + 1 - \sqrt{\hat{x}^2 + \hat{y}^2}) \tag{2.29}$$

with the infinitely differentiable function $H(x)$ defined as

$$H(x) = \int_{-\infty}^{x} \Theta(s)\, \mathrm{d}s \bigg/ \int_{-\infty}^{+\infty} \Theta(s)\, \mathrm{d}s \tag{2.30}$$

and

$$\Theta(s) = \begin{cases} \exp(-1/(1 - |s|^2)), & |s| < 1, \\ 0, & |s| \geqslant 1. \end{cases} \tag{2.31}$$

Then $H(x) = 0$ if $x < -1$ and $H(x) = 1$ if $x > 1$. The constant D is chosen such that the whole body is contained in a circle with diameter D and as centre, the centre of gravity of the body.

The Laplace equation and boundary conditions for the function $\hat{\eta}_{\hat{x}}$ then are transformed into a Poisson equation for $\hat{\psi}_{\hat{x}}$:

$$\nabla^2\hat{\psi}_{\hat{x}} = \nabla^2(\hat{x}H(\tfrac{1}{2}D + 1 - \sqrt{\hat{x}^2 + \hat{y}^2})) \qquad \text{in } \Omega, \tag{2.32}$$

$$\mathbf{n}\cdot\nabla\hat{\psi}_{\hat{x}} = 0 \qquad \text{at } S \cup S_\infty, \tag{2.33}$$

with equivalent expressions for $\hat{\psi}_{\hat{y}}$ and $\hat{\psi}_{\hat{z}}^{M}$.

The only modification necessary in the algorithm for the solution of the Poisson equation used in the vortex method is the change in boundary conditions at the surface S from $\mathbf{n}\cdot\nabla\times\hat{\psi}_{\hat{x}}\hat{\mathbf{e}}_{\hat{z}}$ into $\mathbf{n}\cdot\nabla\hat{\psi}_{\hat{x}}$. The complexity of this change of boundary conditions strongly depends on the choice of the numerical algorithm used for the solution of the Poisson equation. In three dimensions it is necessary to determine six functions $\hat{\eta}$, for all three components of forces and moments. Again it is possible to use the algorithm for the solution of the vector Poisson equation in the vortex method by defining the vector $\hat{\boldsymbol{\eta}}$ as

$$\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{\psi}} - \hat{\mathbf{x}}H(\tfrac{1}{2}D + 1 - |\hat{\mathbf{x}}|), \tag{2.34}$$

where the constant D is chosen such that the whole body is contained inside the sphere with diameter D and its centre at the centre of gravity of the body.

The vector $\hat{\boldsymbol{\psi}}$ then is obtained by solving the following Poisson equation:

$$\nabla^2 \hat{\boldsymbol{\psi}} = \nabla^2(\hat{\mathbf{x}} H(\tfrac{1}{2}D + 1 - |\hat{\mathbf{x}}|)) \qquad \text{in } \Omega, \tag{2.35}$$

$$\mathbf{n} \cdot \nabla \hat{\boldsymbol{\psi}} = 0 \qquad \text{at } S \cup S_\infty, \tag{2.36}$$

with equivalent expressions for the vector $\hat{\boldsymbol{\eta}}^M$, related to the moment.

(iii) The final step in the implementation of the calculation of the forces and moments using equation (2.24) consists of the numerical evaluation of the integrals at the body surface S and the volume integrals over the small neighbourhoods of the vortex blobs. The choice of integration rules should be related to the accuracy of the numerical algorithm used for the determination of the flow field.

Conclusions

The procedure for the calculation of forces and moments in a viscous fluid described by primitive variables can be transformed into a method more suited to vortex methods. The algorithm presents no considerable additional effort because it uses the algorithm for the solution of a Poisson equation which is a basic element of any vortex method.

Acknowledgement

The author wishes to thank Professor A.J. Hermans of the Technical University Delft, The Netherlands, for his advice and interest in this research.

Appendix

Derivation of an expression for the forces and moments due to the acceleration of body or fluid

In this appendix the transformation of the integral containing the fluid acceleration in equation (2.20) into an expression more suited for numerical evaluation, as given by equation (2.24), will be demonstrated. Especially the integration at the surface S_∞ requires some attention. The integrals containing the acceleration of the fluid are calculated in the body-fixed coordinate system ($\hat{\mathbf{e}}_{\hat{x}}$, $\hat{\mathbf{e}}_{\hat{y}}$, $\hat{\mathbf{e}}_{\hat{z}}$). The normal component of the velocity field $\hat{\mathbf{u}}$ at the body surface S with respect to the body-fixed coordinate system then is equal to

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{n}} = -T^{-1}(\boldsymbol{\alpha}_g)\mathbf{U} \cdot \hat{\mathbf{n}}, \tag{3.1}$$

using the no-slip condition at S, equation (2.4). Here the matrix $T(\boldsymbol{\alpha}_g)$ gives the relation between the inertial and body-fixed coordinate system and is defined in equation (2.26).

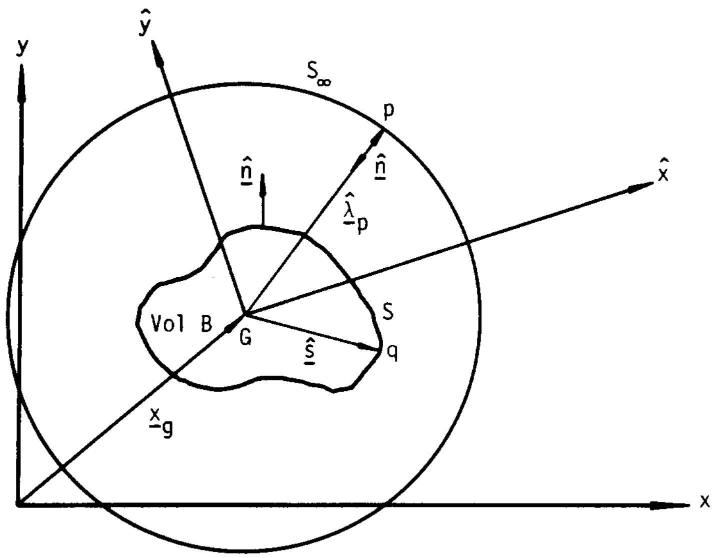

Fig. 2. Definition of the variables in the inertial and body-fixed coordinate systems.

The vector $\boldsymbol{\alpha}_g$ is the rotation axis of the centre of gravity. Here a variable with a ^ always is defined in the body-fixed coordinate system. Analogously, the normal component of the velocity field $\hat{\mathbf{u}}$ at the body surface S_∞ with respect to the body-fixed coordinate system is given by the relation

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{n}} \;=\; -\hat{\mathbf{n}} \cdot (T^{-1}(\boldsymbol{\alpha}_g)\dot{T}(\boldsymbol{\alpha}_g)\hat{\boldsymbol{\lambda}}_p + T^{-1}(\boldsymbol{\alpha}_g)\dot{\mathbf{x}}_g), \tag{3.2}$$

using the condition that the disturbance velocity field approaches a zero value at S_∞. The vector $\hat{\boldsymbol{\lambda}}_p$ is directed from the centre of gravity of the body toward a point p at the surface S_∞, see Figure 2. The relation for the normal component of the velocity field at S_∞ can be further simplified into

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{n}} \;=\; -T^{-1}(\boldsymbol{\alpha}_g)\,\dot{\mathbf{x}}_g \cdot \hat{\mathbf{n}}. \tag{3.3}$$

This result is obtained using the relation

$$T^{-1}(\boldsymbol{\alpha}_g)\dot{T}(\boldsymbol{\alpha}_g) \;=\; -J \tag{3.4}$$

with the matrix J defined as

$$J \;=\; \begin{pmatrix} 0 & -\dot{\alpha}_3 & \dot{\alpha}_2 \\ \dot{\alpha}_3 & 0 & -\dot{\alpha}_1 \\ -\dot{\alpha}_2 & \dot{\alpha}_1 & 0 \end{pmatrix}$$

and $\dot{\boldsymbol{\alpha}}_g = (\alpha_1, \alpha_2, \alpha_3)$ the angular velocity vector.

The first contribution on the right-hand side of equation (3.2) then disappears because

$$-\hat{\mathbf{n}} \cdot T^{-1}(\boldsymbol{\alpha}_g)\dot{T}(\boldsymbol{\alpha}_g)\hat{\boldsymbol{\lambda}}_p \;=\; \hat{\mathbf{n}} \cdot (\dot{\boldsymbol{\alpha}}_g \times \hat{\boldsymbol{\lambda}}_p) \;=\; 0, \tag{3.5}$$

because $\hat{\mathbf{n}}$ and $\hat{\boldsymbol{\lambda}}_p$ point in the opposite direction (see Fig. 2).

The expression for the forces and moments due to the fluid acceleration in the body-fixed coordinate system, as shown in equation (2.20), now can be transformed with the aid of equations (3.1) and (3.3) into

$$\int_{S \cup S_\infty} \hat{\eta} \left(\hat{\mathbf{n}} \cdot \frac{\partial \hat{\mathbf{u}}}{\partial t} \right) \mathrm{d}S \;=\; -\frac{\partial}{\partial t}(T^{-1}(\boldsymbol{\alpha}_g)\mathbf{U}) \cdot \int_S \hat{\mathbf{n}}\hat{\eta}\, \mathrm{d}S - \frac{\partial}{\partial t}(T^{-1}(\boldsymbol{\alpha}_g)\dot{\mathbf{x}}_g) \cdot \int_{S_\infty} \hat{\mathbf{n}}\hat{\eta}\, \mathrm{d}S. \tag{3.6}$$

Here the function $\hat{\eta}$ is one of the functions $\hat{\eta}_{\hat{x}}$, $\hat{\eta}_{\hat{y}}$ or $\hat{\eta}_{\hat{z}}$ for the $\hat{x}$-component of the force etc., or one of the functions $\hat{\eta}^M_{\hat{x}}$, $\hat{\eta}^M_{\hat{y}}$ or $\hat{\eta}^M_{\hat{z}}$ for the components of the moment.

The integration at the surface S presents no special problems, but the integration at the surface S_∞ requires a special treatment, which will be further explained.

The function $\hat{\eta}_p$ at a point p at the surface S_∞ can be expressed in terms of the function $\hat{\eta}_q$ and its boundary condition $\hat{b}_q$ at the body surface S by means of Green's representation theorem:

$$\hat{\eta}_p \;=\; \frac{1}{4\pi}\int_S \left(\hat{\eta}_q\, (\hat{\mathbf{n}}_q \cdot \nabla_q)\left(\frac{1}{R}\right) - \hat{b}_q \left(\frac{1}{R}\right)\right) \mathrm{d}S_q, \tag{3.7}$$

with the function $\hat{b}_q$ equal to the right-hand side of equation (2.17), transformed to the body-fixed coordinate system, and the function R is defined as the distance between the point $\hat{\mathbf{x}}_p$ at the surface S_∞ and $\hat{\mathbf{s}}_q$ at the surface S. The representation of $\hat{\eta}_p$ by means of Green's representation theorem can be further simplified by expanding $1/R$ and $\nabla_q(1/R)$ for large values of the distance between the points $\hat{\mathbf{x}}_p$ and $\hat{\mathbf{s}}_q$.

The integration at the arbitrary surface S_∞ in equation (3.6) then can be transformed into an integration at the surface S, by assuming that the surface S_∞ is a sphere with an infinitely large radius and introducing equation (3.7), using the asymptotic expansion of $1/R$ and $\nabla_q(1/R)$, in equation (3.6) yielding

$$\int_{S_\infty} \hat{\mathbf{n}}\hat{\eta}\, \mathrm{d}S \;=\; -\frac{1}{3}\int_S (\hat{\mathbf{n}}\hat{\eta} - \hat{b}\hat{\mathbf{s}})\, \mathrm{d}S. \tag{3.8}$$

This result is valid for arbitrary three-dimensional bodies. In case of two-dimensional bodies the same result is obtained, however, with the constant 1/3 replaced by 1/2.

Combining the various contributions for the $\hat{x}$, $\hat{y}$ and $\hat{z}$ components the following expression for the forces due to the acceleration of the fluid or body is obtained:

$$\hat{\mathbf{f}}_{\mathrm{acc}} \;=\; \int_{S \cup S_\infty} \hat{\boldsymbol{\eta}} \left(\hat{\mathbf{n}} \cdot \frac{\partial \hat{\mathbf{u}}}{\partial t} \right) \mathrm{d}S \;=\; -B(\dot{T}^{-1}\mathbf{U} + T^{-1}\dot{\mathbf{U}}) + G(\dot{T}^{-1}\dot{\mathbf{x}}_g + T^{-1}\ddot{\mathbf{x}}_g), \tag{3.9}$$

with matrices B and G defined as

$$B_{ij} = \int_S \hat{\eta}_i \hat{n}_j \, \mathrm{d}S, \qquad G_{ij} = \frac{1}{3} \int_S (\hat{\eta}_i \hat{n}_j - \hat{b}_i \hat{s}_j) \, \mathrm{d}S, \tag{3.10}$$

where the indices $i, j = \{1, 2, 3\}$ refer to the $\hat{x}$, $\hat{y}$ or $\hat{z}$ components of the vectors respectively. Here a dot means differentiation with respect to time.

Finally the result as presented in equation (2.24) is obtained by transformation to the inertial coordinate system, viz.

$$\begin{aligned} \mathbf{f}_{\mathrm{acc}} &= T(\boldsymbol{\alpha}_g)\hat{\mathbf{f}}_{\mathrm{acc}} \\ &= B(\dot{\boldsymbol{\alpha}}_g \times \mathbf{U} - \dot{\mathbf{U}}) - G(\dot{\boldsymbol{\alpha}}_g \times \dot{\mathbf{x}}_g - \ddot{\mathbf{x}}_g), \end{aligned} \tag{3.11}$$

where the relation, equation (3.4), for the transformation matrix is used.

Analogous expressions are obtained for the moment by replacing the vectors $\hat{\boldsymbol{\eta}}$ and $\hat{\mathbf{b}}$ by $\hat{\boldsymbol{\eta}}^M$ and $\hat{\mathbf{b}}^M$ respectively.

References

1. Van Oortmerssen, G., van der Vegt, J.J.W. and van Walree, F., Forces on cylinders in oscillatory flow: a comparison of the results of numerical and physical models, *PRADS'87 Conf.*, Trondheim, Norway (1987).
2. Stansby, P.K. and Dixon, A.G., Simulation of flows around cylinders by a Lagrangian vortex scheme, *Appl. Ocean Res.* 5 (1983) 167–178.
3. Chorin, A.J. Numerical study of slightly viscous flow, *J. Fluid Mech.* 57 (1973) 785–796.
4. Milinazzo, F. and Saffman, P.G., The calculation of large Reynolds number two-dimensional flow using discrete vortices with random walk, *J. Comp. Phys.* 23 (1977) 380–392.
5. Beale, J.T. and Majda, A., Rates of convergence for viscous splitting of the Navier–Stokes equations, *Math. of Comp.* 37 (1981) 243–259.
6. Van der Vegt, J.J.W., *A variationally optimized vortex tracing algorithm for three-dimensional flows around solid bodies*, Thesis, Technical University Delft, The Netherlands (1988).
7. Beale, J.T. and Majda, A., Vortex methods I: convergence in three dimensions, *Math. of Comp.* 39 (1982) 1–27.
8. Beale, J.T. and Majda, A., Vortex methods II: higher order accuracy in two and three dimensions, *Math. of Comp.* 39 (1982) 29–52.
9. Anderson, C. and Greengard, C., On vortex methods, *SIAM J. of Num. Anal.* 22 (1985) 413–440.
10. Baden, S.B., Dynamic load balancing of a vortex calculation running on multiprocessors, to be published.
11. Greengard, L. and Rokhlin, V., A fast algorithm for particle simulations, *Research Report* YALEU/DCS/RR-459 (1986).
12. Couët, B., Buneman, O. and Leonard, A., Simulation of three-dimensional incompressible flows with a vortex-in-cell method, *J. Comp. Phys.* 39 (1981) 305–328.
13. Serrin, J., Mathematical principles of classical fluid mechanics, *Encyclopedia of Physics*, Vol. VIII/1, ed.: S. Flügge, Springer-Verlag (1959).
14. Graham, J.M.R., The forces on sharp-edged cylinders in oscillatory flow at low Keulegan-Carpenter numbers, *J. Fluid Mech.* 97 (1980) 331–346.
15. Deffenbaugh, F.D. and Marshall, F.J., Time development of the flow about an impulsively started cylinder, *AIAA Journal* 14 (1976) 908–913.
16. Quartapelle, L. and Napolitano, M., Force and moment in incompressible flows, *AIAA Journal* 21 (1983) 911–913.